On a Theorem of Greuel and Steenbrink

Duco van Straten

To Gert-Martin Greuel on the occasion of his 70th birthday.

Abstract A famous theorem of Greuel and Steenbrink states that the first Betti number of the Milnor fibre of a smoothing of a normal surface singularity vanishes. In this paper we prove a general theorem on the first Betti number of a smoothing that implies an analogous result for weakly normal singularities.

Keywords: Singularities, topology of smoothings, weakly normal spaces.
2010 Mathematics Subject Classification: 14B07, 32S25, 32S30.

Introduction

By a *singularity* we usually mean a germ $(X,p) \subset (\mathbb{C}^N,p)$ of a complex space, but in order to study its topology, it is customary to pick an appropriate contractible Stein representative X of the germ in question. By a *deformation* of the singularity (X,p) over $(S,0)$ we understand a pull-back diagram of the form

$$\begin{array}{ccc} X & \hookrightarrow & \mathscr{X} \\ \downarrow & & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array}$$

where the map $f : \mathscr{X} \longrightarrow S$ is flat and all spaces are appropriate representives of the corresponding germs. We say the deformation is a *smoothing* if the general fibre $X_t = f^{-1}(t)$, $t \in S$, is smooth, in which case we say that X_t is the *Milnor-fibre* of the smoothing under consideration. In the classical case of isolated hypersurface singularities, Milnor [4] has shown that this fibre has the homotopy type of a bouquet of

Duco van straten
Institut für Mathematik, Johannes Gutenberg Universität, 55128 Mainz, Germany. e-mail: straten@mathematik.uni-mainz.de

spheres of dimension equal to the complex dimension of $(X,0)$. It is of considerable interest to find topological properties of Milnor fibres in more general situations.

In the paper [3] of Greuel and Steenbrink one find an overview of some of the basic results and in particular a proof of the following result that was conjectured by Wahl in [10].

Theorem 1: *Let $\mathscr{X} \xrightarrow{f} S$ be a smoothing of an isolated normal singularity $X := f^{-1}(0)$ and let $X_t := f^{-1}(t)$, $t \neq 0$, denote its Milnor fibre. Then:*

$$b_1(X_t) := \dim_{\mathbb{C}} H^1(X_t, \mathbb{C}) = 0 \,.$$

Note that for a normal surface singularity the fundamental group $\pi_1(X_t)$ or even $H_1(X_t)$ of the Milnor-fibre need not to be trivial.

When one looks for a similar simple statement for non-isolated singularities, one soon runs into difficult problems. In [11] Zariski described two types of 6-cuspidal sextics in $\mathbb{P}^2$, for which the complements have different fundamental groups. By taking the cone over such a curve, we get a surface in $\mathbb{C}^3$, whose Milnor fibre appears as the cyclic six-fold cover of the complement of this curve. Its first Betti number depends on the position of the cusps: when they are on a conic, then $b_1(X_t) = 2$, when they are not, then $b_1(X_t) = 0$, [2]. This shows that the first Betti number b_1 is a subtle invariant.

In this paper we will show the following general theorem.

Theorem 2: *Let $\mathscr{X} \xrightarrow{f} S$ a smoothing of a reduced and equidimensional germ (X,p). Let $X_t = f^{-1}(t), t \neq 0$, its Milnor fibre. Let $X^{[0]} = \sqcup_{i=1}^{r} X_i$, where the X_i are the ireducible components of X. Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl(\mathscr{X},p)$ be the map that associated to a divisor supported on X its class in the local class group. Then one has:*

1. $b_1(X_t) \geq \mathrm{rank}(\ker\gamma) - 1$.
2. *When X is weakly normal, then one has equality:*

$$b_1(X_t) = \mathrm{rank}(\ker\gamma) - 1.$$

In this case the action of the monodromy in $H^1(X_t)$ is trivial.

We spell out two useful corollaries of this general result:

Corollary 1: *If (X,p) is a hypersurface singularity with r irreducible components, then $b_1(X_t) \geq r-1$, with equality in the case that (X,p) is weakly normal.*

Corollary 2: *Let $\mathscr{X} \xrightarrow{f} S$ be a smoothing of a reduced, equidimensional and weakly normal space germ $X = f^{-1}(0)$ and let $X_t := f^{-1}(t)$, $t \neq 0$ denote its Milnor fibre. Then*

$$b_1(X_t) \leq r-1.$$

where r denotes the number of irreducible components of X. For a hypersurface equality holds.

Recall that a complex space germ X is called *weakly normal*, if every function that is continuous and holomorphic outside the singular set of X is in fact holomorphic on all of X. The union of coordinate axis in $\mathbb{C}^n$ is the unique weakly normal curve singularity with multiplicity n and a weakly normal surface has such a curve singularity as generic transversal type. If in addition X is Cohen-Macaulay, then also the converse holds. In particular, the cone over a plane curve $\Gamma \subset \mathbb{P}^2$ is weakly normal precisely when Γ has only ordinary double points. In this case the first Betti number is independent of the exact position of the double points: one has $b_1(X_t) = r-1$, where r denotes the number of irreducible components of Γ.

Our proof of **Theorem 2** will given in the following sections and runs along the lines of the paper [3].

1 Embedded resolution

Let X be a fixed contractible Stein representative of a *reduced and equidimensional* germ (X,p). We consider a smoothing over a smooth curve germ S:

$$\begin{array}{ccc} X & \hookrightarrow & \mathscr{X} \\ \downarrow & & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array}$$

The following fact is well-known:

Lemma 1: *The total space of the smoothing $\mathscr{X}$ of a reduced and equidimensional germ is* normal.

proof: As we are dealing with a smoothing over a smooth curve germ, the singular locus $Sing(\mathscr{X})$ of $\mathscr{X}$ is a subset of the singular locus $\Sigma := Sing(X)$ of X, so this set is of codimension ≥ 2 in $\mathscr{X}$. Furthermore, as X is reduced we have $\mathrm{depth}_\Sigma(X) \geq 1$, so that from the flatness of the family we obtain $\mathrm{depth}_\Sigma(\mathscr{X}) \geq 2$, hence $\mathscr{X}$ is normal. $\Diamond$

To study the Milnor fibre $X_t := f^{-1}(t), t \neq 0$ of a smoothing, we will make use of an *embedded resolution* of X in $\mathscr{X}$. By Hironaka's theorem there exists an appropriate sequence of blow-ups that produces a space $\mathscr{Y}$ together with a proper map

$$\pi : \mathscr{Y} \longrightarrow \mathscr{X}$$

which has the following properties:

- $\mathscr{Y}$ is smooth.
- $Y := (f \circ \pi)^{-1}(0)$ is a normal crossing divisor.
- $\pi : \mathscr{Y} \setminus \pi^{-1}(\Sigma) \longrightarrow X \setminus \Sigma$ is an isomorphism.

By the semi-stable reduction theorem we may and will assume, after first performing an appropraite finite base-change on S, that the divisor Y in addition is *reduced*.

The divisor Y can be decomposed into two parts:

a) $\widetilde{X}$, the strict transform of X.
b) $F = \cup_i F_i$, a non-compact divisor, mapping properly onto Σ.

The component of F can be grouped further according to the stratum of Σ they map to, but in this paper will not need to do so.

1.1 Leray sequence

Via the map π the fibre $Y_t := (f \circ \pi)^{-1}(t) \subset \mathscr{Y}$ is isomorphic to the Milnor fibre X_t

$$\pi : Y_t \xrightarrow{\simeq} X_t .$$

In a semi-stable family the divisor Y is reduced, so that the space Y_t "passes along every component of Y just once". More precisely, one can find a contraction map

$$c : Y_t \longrightarrow Y$$

of the Milnor fibre Y_t onto the special fibre Y, which is an isomorphism on the preimage of the subset of regular points of Y, see for example [1]. One can try to compute the cohomology of Y_t using the Leray spectral sequence for the map c. It is easy to verify from the local model of the maps f and c that

$$c_*(\mathbb{Z}_{Y_t}) = \mathbb{Z}_Y ,$$

$$R^1 c_*(\mathbb{Z}_{Y_t}) = \mathbb{Z}_{Y^{[0]}} / \mathbb{Z}_Y .$$

Here $Y^{[0]} := \sqcup_i Y_i$, denotes the disjoint union of the irreducible components of Y, which naturally maps to Y. Via this map we consider the constant sheaf $\mathbb{Z}_{Y^{[0]}}$ as a sheaf on Y. Indeed, the fibre of c over a point in Y is homotopy equivalent to a real torus of dimension equal $k-1$, where k is the number irreducible components of Y passing through the point. From this description one obtains

$$(\mathbb{Z} \simeq) H^0(Y) \xrightarrow{\simeq} H^0(Y_t)$$

and the beginning of an exact sequence of cohomology groups (always with $\mathbb{Z}$-coefficients, unless stated otherwise):

Leray sequence:

$$(1): \quad 0 \longrightarrow H^1(Y) \longrightarrow H^1(Y_t) \longrightarrow H^0(\mathbb{Z}_{Y^{[0]}}/\mathbb{Z}_Y) \longrightarrow H^2(Y) \longrightarrow \cdots$$

Note that we also have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_Y \longrightarrow \mathbb{Z}_{Y^{[0]}} \longrightarrow \mathbb{Z}_{Y^{[0]}}/\mathbb{Z}_Y \longrightarrow 0$$

of sheaves on Y. From the associated long exact sequence of cohomology groups we obtain the beginning of an exact sequence

$$(2): \quad 0 \longrightarrow H^0(Y) \longrightarrow H^0(Y^{[0]}) \longrightarrow H^0(\mathbb{Z}_{Y^{[0]}}/\mathbb{Z}_Y) \longrightarrow H^1(Y) \longrightarrow \cdots$$

1.2 Two further sequences and a diagram

There are two further exact sequences in which $H^1(Y_t)$ appears:

Milnors Wang-sequence (see [4], p.67):

$$(3): \quad 0 \longrightarrow H^0(Y_t) \longrightarrow H^1(B) \longrightarrow H^1(Y_t) \xrightarrow{h^*-Id} H^1(Y_t) \longrightarrow \cdots$$

Here $B := \mathscr{X} \setminus X$, the total space of the Milnor fibration over $S \setminus \{0\}$ and h^* denotes the cohomological monodromy transformation.

The cohomology sequence of the pair: $\mathscr{Y} \setminus Y \hookrightarrow \mathscr{Y}$ reads:

$$\cdots \longrightarrow H^1(Y) \longrightarrow H^1(\mathscr{Y} \setminus Y) \longrightarrow H^2(\mathscr{Y}, \mathscr{Y} \setminus Y) \longrightarrow H^2(Y) \longrightarrow \cdots$$

We use the isomorphism

$$\mathscr{Y} \setminus Y \xrightarrow{\simeq} \mathscr{X} \setminus X = B.$$

Furthermore, from the homoptopy equivalence $\mathscr{Y} \simeq Y$ and the Lefschetz isomorphism we obtain

$$H^1(\mathscr{Y}, \mathscr{Y} \setminus Y) = 0, \quad H^2(\mathscr{Y}, \mathscr{Y} \setminus Y) \simeq H^0(Y^{[0]}),$$

so the sequence of the pair is seen to reduce to the exact sequence

$$(4): \quad 0 \longrightarrow H^1(Y) \longrightarrow H^1(B) \xrightarrow{\alpha} H^0(Y^{[0]}) \xrightarrow{\beta} H^2(Y) \longrightarrow \cdots$$

We will describe the map β in detail in section 4.

These four sequences fit into a single commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H^0(Y_t) & \simeq & H^0(Y) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(B) & \xrightarrow{\alpha} & H^0(Y^{[0]}) & \xrightarrow{\beta} & H^2(Y) \longrightarrow \cdots \\
 & & \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(Y_t) & \longrightarrow & H^0(\mathbb{Z}_{Y^{[0]}}/\mathbb{Z}_Y) & \longrightarrow & H^2(Y) \longrightarrow \cdots \\
 & & & & \downarrow h^* - Id & & \downarrow & & \\
 & & & & H^1(Y_t) & & H^1(Y) & &
\end{array}
$$

Lemma 2: *In the above situation we have:*

1. $\operatorname{rank} H^1(B) = \operatorname{rank} H^1(Y) + \operatorname{rank} (\ker \beta)$.
2. $\operatorname{rank} H^1(Y_t) \geq \operatorname{rank} H^1(B) - 1$.
3. If $H^1(Y) = 0$*, then the monodromy acts trivially on* $H^1(Y_t)$.
4. If $H^1(Y) = 0$*, then* $\operatorname{rank} H^1(Y_t) = \operatorname{rank} H^1(B) - 1$.

proof: First recall that we have $\operatorname{rank} H^0(Y) = 1 = \operatorname{rank} H^0(Y_t)$. From the first exact row of the diagram, coming sequence (4), we read off the first statement. The first column of the diagram, coming from sequence (3), gives the second statement. If $H^1(Y)$ is assumed to be zero, the diagram simplifies, and a diagram chase learns that the map $H^1(B) \longrightarrow H^1(Y_t)$ is surjective, so that $h_* - Id$ is the zero map on $H^1(Y_t)$, which is the third statement. The last statement then follows by looking again at the first column of the diagram. ◊

We now study the parts $H^1(Y)$ and $\ker \beta$ separately.

2 The group $H^1(Y)$

If X is a plane curve singularity, then it is easy to determine $\operatorname{rank} H^1(Y)$. The result is

$$\operatorname{rank} H^1(Y) = 2g + b \,,$$

where g is the sum of the genera of the compact components of Y and b is the number of cycles in the dual graph of Y. These numbers g and b are in fact invariants of the limit Mixed Hodge Structure on $H^1(X_t)$; one has $b = \dim_0^W Gr_F^1 H^1(X_t)$, $g = \dim Gr_1^W Gr_F^1 H^1(X_t)$, see [7]. By taking $X \times \mathbb{C}$ we obtain in a trivial way examples of irreducible surfaces with arbitrary high Betti number. Only in the case that X is an ordinary double point, one has $H^1(Y) = 0$. It turns out that in general it is exactly the *weak normality* of X that forces $H^1(Y)$ to vanish.

Proposition 1: *Let $\mathscr{X} \stackrel{f}{\longrightarrow} S$ a flat deformation of a weakly normal space $X = f^{-1}(p)$. Let $\mathscr{Y} \stackrel{\pi}{\longrightarrow} \mathscr{X}$ a map such that*

1. $\mathscr{Y} \setminus \pi^{-1}(\Sigma) \stackrel{\simeq}{\longrightarrow} \mathscr{X} \setminus \Sigma$, $\Sigma := Sing(X)$ is an isomorphism.
2. $\pi_* \mathscr{O}_{\mathscr{Y}} \simeq \mathscr{O}_{\mathscr{X}}$.

Then one has

$$R^1 \pi_* \mathscr{O}_{\mathscr{Y}} = 0 \ .$$

proof: The argument is basically the same as in [3]. First look at the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathscr{Y}} \stackrel{t\cdot}{\longrightarrow} \mathscr{O}_{\mathscr{Y}} \longrightarrow \mathscr{O}_Y \longrightarrow 0.$$

Here t is a local parameter on S and $t\cdot$ is the map obtained from multiplication by t. The space Y is defined by the equation $t = 0$; it is the fibre over $0 \in S$. When we take the direct image of this sequence under π, we obtain a diagram

$$\begin{array}{ccccccccccc} 0 \longrightarrow & \mathscr{O}_{\mathscr{X}} & \stackrel{t\cdot}{\longrightarrow} & \mathscr{O}_{\mathscr{X}} & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 & & & \\ & \downarrow & & \downarrow & & \downarrow & & & & & \\ 0 \longrightarrow & \pi_* \mathscr{O}_{\mathscr{Y}} & \stackrel{t\cdot}{\longrightarrow} & \pi_* \mathscr{O}_{\mathscr{Y}} & \longrightarrow & \pi_* \mathscr{O}_Y & \longrightarrow & R^1 \pi_* \mathscr{O}_{\mathscr{Y}} & \stackrel{t\cdot}{\longrightarrow} & R^1 \pi_* \mathscr{O}_{\mathscr{Y}} & \cdots \end{array}$$

By assumptiom, the natural map $\mathscr{O}_{\mathscr{X}} \longrightarrow \pi_* \mathscr{O}_{\mathscr{Y}}$ is an isomorphism. From this it follows that the sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \pi_* \mathscr{O}_Y \longrightarrow R^1 \pi_* \mathscr{O}_{\mathscr{Y}} \stackrel{t\cdot}{\longrightarrow} R^1 \pi_* \mathscr{O}_{\mathscr{Y}} \longrightarrow \cdots$$

is also exact. We claim that we also have an isomorphism

$$\mathscr{O}_X \simeq \pi_* \mathscr{O}_Y \ .$$

Note that it follows from condition 2) that the fibres of π are *connected*. Consider a section $g \in \pi_* \mathscr{O}_Y$, or, what amounts to the same, a function on Y. As the π-fibres are compact and connected, this function is *constant* along the π-fibres. Hence g can be considered as a *continuous* function on X, which is holomorphic on $Y \setminus \pi^{-1}(\Sigma) = X \setminus \Sigma$. Because we assumed X to be weakly normal, it follows that g is holomorphic on X: $g \in \mathscr{O}_X$. So the map $\mathscr{O}_X \hookrightarrow \pi_* \mathscr{O}_Y$ is indeed an isomorphism. Because the map π is an isomorphism outside Σ, the coherent sheaf $R^1 \pi_* \mathscr{O}_{\mathscr{Y}}$ has as support a set contained in Σ. But the last exact sequence now tells us that $t\cdot$ acts injective on $R^1 \pi_* \mathscr{O}_{\mathscr{Y}}$. As t vanishes on $\Sigma \subset X$ we conclude that $R^1 \pi_* \mathscr{O}_{\mathscr{Y}} = 0$. $\Diamond$

For any weakly normal surface singularity (X, p) one can construct an *improvement* $\pi : Y \longrightarrow X$, which is an isomorphism over $X \setminus \{p\}$ and which has only certain basic weakly normal singularities, called partition singularities. Such an improvement plays a role analogous to that of the resolution for normal singularities. Weakly rational singularities are defined by the vanishing of $R^1 \pi_* \mathscr{O}_Y$. For more details we refer to [9]. Proposition 1 implies the following statement:

Let C be a weakly normal curve singularity and X the total space of a flat deformation $X \longrightarrow S$ of C. Then X is weakly rational.

This follows from the above proposition by applying it to $X = C$, $\mathscr{X} = X$, $\mathscr{Y} = Y$. Note that for Proposition 1 we did not assume $\mathscr{Y}$ to be smooth.

We return to the general situation of a smoothing of a reduced equidimensional space X.

Proposition 2: *With the same notations as before, we have for the smoothing of a reduced, equidimension space X the following implication:*

$$X \quad \text{weakly normal} \quad \Longrightarrow \quad H^1(Y) = 0\ .$$

proof: The embedded resolution map $\mathscr{Y} \longrightarrow \mathscr{X}$ clearly satisfies the condition 1) of Proposition 1. It follows from Lemma 1 that $\mathscr{X}$ is normal, hence $\mathscr{O}_{\mathscr{X}} \xrightarrow{\simeq} i_*\mathscr{O}_{\mathscr{X}\setminus\Sigma}$. Because $\mathscr{Y} \setminus \pi^{-1}(\Sigma) \longrightarrow X \setminus \Sigma$ is an isomorphism, it follows that $\pi_*\mathscr{O}_{\mathscr{Y}} = \mathscr{O}_{\mathscr{X}}$. Hence also the second condition of Proposition 1 is fulfulled, so we can conclude that $R^1\pi_*\mathscr{O}_{\mathscr{Y}} = 0$, in other words we get

$$H^1(\mathscr{O}_{\mathscr{Y}}) = 0.$$

From the exponential sequence on $\mathscr{Y}$

$$0 \longrightarrow \mathbb{Z}_{\mathscr{Y}} \longrightarrow \mathscr{O}_{\mathscr{Y}} \longrightarrow \mathscr{O}^*_{\mathscr{Y}} \longrightarrow 0\ ,$$

the similar sequence for $\mathscr{X}$ and the fact the $\mathscr{O}_{\mathscr{X}} \simeq \pi_*\mathscr{O}_{\mathscr{Y}}$ it follows that

$$H^1(\mathscr{Y}, \mathbb{Z}_{\mathscr{Y}}) = 0$$

As $\mathscr{Y}$ contracts onto Y we have $H^1(Y, \mathbb{Z}) = 0$. ◊

3 The kernel of β

In the big diagram there was a map β

$$H^0(Y^{[0]}, \mathbb{Z}) \xrightarrow{\beta} H^2(\mathscr{Y}, \mathbb{Z})(= H^2(Y, \mathbb{Z}))$$

This map works as follows: Elements of the first group can be considered as *divisors* $\sum n_i Y_i$ supported on Y. Each such divisor determines a *line bundle* $\mathscr{O}(\sum_i n_i Y_i)$. Then one has

$$\beta(\sum n_i Y_i) = c_1(\mathscr{O}(\sum n_i Y_i)).$$

So the map β factorises over the map ψ which associates to a divisor its line bundle. From the exponential sequence on $\mathscr{Y}$ we obtain the following diagram:

$$\begin{array}{ccccccc} & & H^0(Y^{[0]}) & = & H^0(Y^{[0]}) & & \\ & & \downarrow \psi & & \downarrow \beta & & \\ \cdots \longrightarrow H^1(\mathscr{O}_{\mathscr{Y}}) & \longrightarrow & H^1(\mathscr{O}^*_{\mathscr{Y}}) & \longrightarrow & H^2(\mathscr{Y},\mathbb{Z}) & \longrightarrow & \cdots \end{array}$$

One immediately obtains:

Lemma 3: rank $(\ker\beta) \geq$ rank $(\ker\psi)$. *If* $H^1(\mathscr{O}_{\mathscr{Y}}) = 0$, *then* $\ker\psi = \ker\beta$. $\diamondsuit$

Definition: *Let* $(\mathscr{X}, p)$ *be a germ of a normal analytic space. The* local class group *is defined as*

$$Cl(X,p) := We(X,p)/Ca(X,p)\ .$$

Here $We(X,p)$ *is the free abelian group spanned by the (germs of) Weil divisors on* X *and* $Ca(X,p)$ *the subgroup spanned by the (germs of) Cartier divisors.*

Lemma 4: *With the notations of section 2 there is a diagram with exact rows and columns:*

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \ker\psi & \xrightarrow{\simeq} & \ker\gamma & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(F^{[0]}) & \longrightarrow & H^0(Y^{[0]}) & \longrightarrow & H^0(X^{[0]}) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \psi & & \downarrow \gamma & & \\ 0 & \longrightarrow & H^0(F^{[0]}) & \longrightarrow & H^1(\mathscr{O}^*_{\mathscr{Y}}) & \longrightarrow & Cl(\mathscr{X},p) & \longrightarrow & 0 \end{array}$$

Here $F^{[0]} := Y^{[0]} \setminus \widetilde{X}$ *is the disjoint union of the divisors* F *of section 1 and* $X^{[0]} = \sqcup_{i=1}^r X_i$, *where the* X_i *are the irreducible components of* X. *The maps are the canonical ones.*

proof: The surjection $H^1(\mathscr{O}^*_{\mathscr{Y}})$ (=$R^1\pi_*\mathscr{O}_{\mathscr{Y}}$) to the local class group works as follows: by pulling back a Weil divisor on $\mathscr{X}$ we obtain a Cartier divisor on $\mathscr{Y}$ (and hence a line bundle) that maps back to the original Weil divisor on $\mathscr{X}$, as the map π is a modification in codimension ≥ 2 (cf. [5]). The main point is to show that the kernel of the map $H^1(\mathscr{O}^*_{\mathscr{Y}}) \longrightarrow Cl(\mathscr{X},p)$ is precisely $H^0(F^{[0]})$, or what amounts to the same, that $\ker\psi = \ker\gamma$. Let $A = \sum n_i Y_i$ be in the kernel of ψ. We may assume that $n_i \geq 0$. Hence there is a function $g \in H^0(\mathscr{O}_{\mathscr{Y}})$ with $(g) = A$. By the normality of $\mathscr{X}$ we have $\mathscr{O}_{\mathscr{X}} = \pi_*\mathscr{O}_{\mathscr{Y}}$, so g can be considered as a holomorphic function on $\mathscr{X}$, having as divisor on $\mathscr{X}$ just the image of that part of of A that does not involve the divisors of $F^{[0]}$. This gives the map $\ker\psi \longrightarrow \ker\gamma$. This map is injective, because if the divisor of g (on $\mathscr{X}$) would be zero, g would be a unit, hence $A = 0$. Surjectivity of $\ker\psi \longrightarrow \ker\gamma$ can be shown as follows: $A = \sum_i n_i X_i$ is an element in $\ker\gamma$ if it is the divisor on $\mathscr{X}$ of some function $g \in \mathscr{O}_{\mathscr{X}}$. The divisor of $g \circ \pi$ is a Cartier divisor on $\mathscr{Y}$ supported on Y, so produces an element of $\ker\psi$ mapping to A. $\diamondsuit$

The use of Lemma 4 is that it allows us to get rid of $\ker\psi$, that depends on the global object $\mathscr{Y}$ over which we have not much control, and replace it with the map

$$\gamma : H^0(X^{[0]}) \longrightarrow Cl(\mathscr{X}, p)$$

that maps each irreducible component of X_i to its class of the corresponding Weil-divisor on $\mathscr{X}$.

There is one further issue: in section 2 we first performed a a base change to arrive at a semi-stable family. The following lemma shows that kernel of the map γ is essentially independent of base change.

Lemma 5: *Consider a normal space $\mathscr{X}$ and a reduced principal divisor $X \subset \mathscr{X}$. Let $X^{[0]} = \sqcup_{i=1}^{r} X_i$, where the X_i are the irreducible components of X. Let $\widetilde{\mathscr{X}}$ be obtained from $\mathscr{X}$ by taking a d-fold cyclic covering ramified along X.*
Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl(\mathscr{X}, p)$ and $\widetilde{\gamma} : H^0(X^{[0]}) \longrightarrow Cl(\widetilde{\mathscr{X}}, p)$ be the maps discussed above. Then

$$\operatorname{rank}(\ker\gamma) = \operatorname{rank}(\ker\widetilde{\gamma}).$$

proof: Let $\phi : \widetilde{\mathscr{X}} \longrightarrow \mathscr{X}$ be the cyclic d-fold covering map. We consider the irreducible components X_i as divisors both on $\mathscr{X}$ and on $\widetilde{\mathscr{X}}$ If $A = \sum_i n_i X_i \in \ker\gamma$, then $A = (g)$ for some $g \in \mathscr{O}_{\mathscr{X}}$. On the covering space $\widetilde{\mathscr{X}}$ the function $g \circ \phi$ now has $d \cdot A$ as divisor. Conversely, if $B = \sum_i m_i X_i \in \ker\widetilde{\gamma}$, then $B = (h)$ for some $h \in \mathscr{O}_{\widetilde{\mathscr{X}}}$. Then the norm $N(h) \in \mathscr{O}_{\mathscr{X}}$ has $d \cdot B$ as divisor. $\diamondsuit$

4 Proof of Theorem 2

In the introduction we formulated the following theorem.

Theorem 2: Let $\mathscr{X} \xrightarrow{f} S$ a smoothing of a reduced equidimensional germ (X, p). Let $X_t = f^{-1}(t), t \neq 0$, its Milnor fibre. Let $X^{[0]} = \sqcup X_i$, where the X_i are the ireducible components of X. Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl(\mathscr{X}, p)$ be the map that associated to a divisor supported on X its class in the local class group. Then one has:

1. $b_1(X_t) \geq \operatorname{rank}(\ker\gamma) - 1$.
2. When X is weakly normal, then one has equality:

$$b_1(X_t) = \operatorname{rank}(\ker\gamma) - 1.$$

In this case the action of the monodromy is trivial.

proof: From Lemma 2, 1) and 2) we get:

$$\text{rank}\, H^1(Y_t) \geq \text{rank}\, H^1(B) - 1 = \text{rank}\, H^1(Y) + \text{rank}\,(\ker \beta) - 1 \geq \text{rank}\,(\ker \beta) - 1$$

Furthermore, from Lemma 3 we have

$$\text{rank}\,(\ker \beta) \geq \text{rank}\,(\ker \psi).$$

From Lemma 4 and 5, the number on the right hand side is the same as

$$\text{rank}(\ker \gamma),$$

so that we get

$$\text{rank}\, H^1(Y_t) \geq \text{rank}(\ker \gamma) - 1$$

which is the first statement of the theorem. For equality it is necessary that $H^1(Y) = 0$ and by Lemma 3 the equality $\text{rank}\,(\ker \beta) = \text{rank}\,(\ker \psi)$ is implied by the vanishing of $H^1(\mathscr{O}_{\mathscr{Y}})$. If we are considering a smoothing of a weakly normal space, this follows from Proposition 2 and 1, respectively. The triviality of the monodromy is Lemma 2, 3). ♢

In particular, when X is a hypersurface, or more generally, if $Cl(\mathscr{X}, p)$ is finite, then $\text{rank}(\ker \gamma)$ is equal to the number r of irreducible components of X, so $b_1(X_t) \geq r - 1$, with equality in the weakly normal case.

Remark: For a hypersurface germ X in $\mathbb{C}^3$ with a complete intersection as singular locus and transversal type A_1, it is known that the first Betti number is zero or one, see [6], [8]. So the number of irreducible components of X is one or two. To put it in another way, the singular locus of a weakly normal hypersurface in $\mathbb{C}^3$ which has more than three components is *never* a complete intersection.

Question: J. Stevens has shown that all degenerate cusps are smoothable. What is the first Betti number for these smoothings? Is the first Betti number an invariant of X? Maybe not, but I do not have computed any non-trivial example. This seems to be an interesting topic for further investigations.

Acknowledgement: The basis of the above text is part of my PhD thesis [9], but the results were never properly published. For this version only minor cosmetic changes have been made. I thank D. Siersma for asking me about the result and the idea of writing it up as a contribution to the volume on occasion of Gert-Martins 70th birthday.

References

1. H. Clemens, *Degeneration of Kähler manifolds*, Duke Math. J. **44** (1977), no. 2, 215 - 290.

2. H. Esnault, *Fibre de Milnor d' un cone sur une courbe plane singulière*, Invent. Math. **68** (1982), no. 3, 477 - 496.
3. G.-M. Greuel, J. Steenbrink, *On the topology of smoothable singularities*, in: Singularities, Part 1 (Arcata, Calif., 1981), 535 - 545, Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, R.I., 1983.
4. J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. **61**, Princeton University Press, Princeton, N.J., University of Tokyo Press, Tokyo 1968.
5. D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. No. **9**, 1961, 5 - 22.
6. D. Siersma, *Singularities with critical locus a 1-dimensional complete intersection and transversal type A1.*, Topology Appl. **27** (1987), no. 1, 51 - 73.
7. J. Steenbrink, *Mixed Hodge structures associated with isolated singularities*, Singularities, Part 2 (Arcata, Calif., 1981), 513 - 536, Proc. Sympos. Pure Math., **40**, Amer. Math. Soc., Providence, RI, 1983.
8. D. van Straten, *On the Betti numbers of the Milnor fibre of a certain class of hypersurface singularities*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), 203 - 220, Lecture Notes in Math. **1273**, Springer, Berlin, 1987.
9. D. van Straten, *Weakly normal surface singularities and their improvements*, Thesis, Leiden 1987.
10. J. Wahl, *Smoothings of normal surface singularities*. Topology **20** (1981), no. 3, 219 - 246.
11. O. Zariski, *On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve*, Amer. J. Math. **51** (1929), no. 2, 305 - 328.